Math 249 Lecture 25 Notes

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1 Species of Graphs

1.1 The species of connected graphs

Let $G(S) = \{\text{graphs with vertex set } S\}$ be the species of graphs. The generating function weighting the edges is

$$G(x;t) = \sum_{n=0}^{\infty} (1+t)^{\binom{n}{2}} \frac{x^n}{n!}.$$

Let $G_C(S) = \{\text{connected graphs with vertex set } S\}$ be the species of connected graphs. By convention, we say that an empty graph is not connected. Then the isomorphism

$$G \cong E \circ G_C$$

is the statement that a graph is the union of its connected components. This gives us that

$$G(x;t) = e^{G_C(x;t)}$$

$$G_C(x;t) = \log(G(x;t)) = \log\left(\sum_{n=0}^{\infty} (1+t)^{\binom{n}{2}} \frac{x^n}{n!}\right)$$

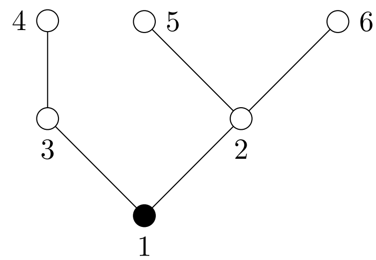
This might seem messy, but it is actually pretty easy to compute, especially by computer. Let's write out a few terms:

$$G_C(x;t) = x + t\frac{x^2}{2!} + (3t^2 + t^3)\frac{x^3}{3!} + (16t^3 + 15t^4 + 6t^5 + t^6)\frac{x^4}{4!} + \cdots$$

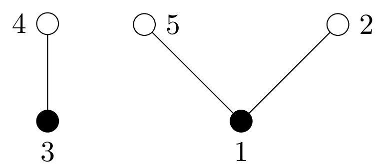
In each term x^n, the coefficient of the least power of t is the number of trees on n vertices. This is because a tree is the connected graph with the fewest edges (if you can remove an edge and stay connected, then your graph has a cycle).

1.2 Rooted trees and Cayley's formula

Definition 1.1. A *rooted* tree is a tree graph with a distinguished vertex called the root.



Definition 1.2. A *rooted forest* is a graph where the connected components are all rooted trees.



Theorem 1.1 (Cayley's formula)**.** *There are n^{n-2} trees on n labeled vertices.*

Proof. Cayley's formula is equivalent to saying that there are n^{n-1} rooted trees. Define the species $T(S) = \{\text{rooted trees on } S\}$. The species $F(S) = \{\text{rooted forests on } S\}$ satisfies the isomorphism

$$F \cong E \circ T.$$

To get rooted trees on a set S, we can pick a vertex in S, make it the root, and make its neighbors the roots of the components in a rooted forest. Denoting X_1 as the indicator species of 1, this gives us the species isomorphism

$$T \cong X_1 F \cong X_1(E \circ T),$$

which gives us the identity

$$T(x) = xe^{T(x)}.$$

Rearranging this, we get that

$$T(x)e^{-T(x)} = x,$$

which says that $T(x)$ is the compositional inverse of xe^{-x}.[1]

[1] At this step, we can the find the coefficients using the Lagrange inversion formula, but instead we will deduce the Lagrange inversion formula as a generalization of this story.

We want to show that

$$T(x) = \sum_{n=1}^{\infty} n^{n-1} \frac{x^n}{n!}.$$

Let's say we have guessed the answer. This seems very similar to

$$\sum_{n=0}^{\infty} n^n \frac{x^n}{n!},$$

which is the exponential generating function for $M(S) = \{\text{maps } S \to S\}$. In fact, the generating functions satisfy

$$xT'(x) = M(s) - 1,$$

so proving this identity will be sufficient for the proof.

Write a function $S \to S$ as a directed graph. If we start at any vertex and follow arrows, we will eventually get to a vertex in a cycle. So M is a composite species

$$M \cong P \circ T.$$

Since $P(x) = \frac{1}{1-x}$, this gives us that

$$M(x) = \frac{1}{1 - T(x)}.$$

Differentiating $T(x) = xe^{T(x)}$, we get

$$T'(x) = e^{T(x)} + xe^{T(x)}T'(x) = e^{T(x)} + T(x)T'(x).$$

$$(1 - T(x))T'(x) = e^{T(x)}$$

$$(1 - T(x))xT'(x) = xe^{T(x)} = T(x).$$

So we get that

$$xT'(x) = \frac{T(x)}{1 - T(x)} = \frac{1}{1 - T(x)} - 1 = M(x) - 1,$$

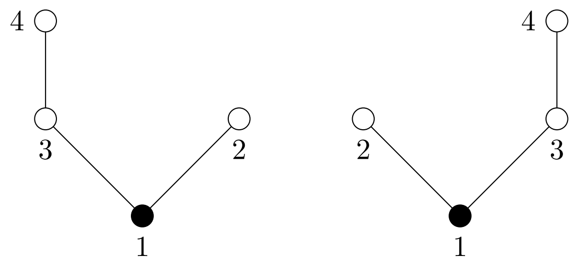
which is what we wanted. So

$$t_n = n^{n-1}. \qquad \square$$

1.3 Plane trees

Definition 1.3. A *plane tree* is a tree where there is a linear ordering on the children of each vertex.

Example 1.1. The following two rooted plane trees are not isomorphic as plane trees, even though they are isomorphic as trees:



Let $T_p(S) = \{\text{plane trees on } S\}$ be the species of plane trees. Plane trees have "no" automorphisms, so

$$T_p(x) = \sum_n t_p(n) \frac{x^n}{n!} = \sum_n c_n x^n,$$

where c_n is the number of unlabelled plane trees on n vertices.

If we take a rooted plane tree and take out the root, we get a forest of rooted plane trees but linearly ordered. This produces the species isomorphism

$$T_p \cong X(L \circ T_p).$$

The corresponding identity fo generating functions is

$$T_p(x) = x \frac{1}{1 - T_p(x)},$$

which gives us the quadratic equation

$$T_p(x)^2 - T_p(x) + x = 0.$$

Solve this equation to get

$$T_p(x) = \frac{1 \pm \sqrt{1 - 4x}}{2}.$$

Which solution is the correct one? We need to have a 0 constant term (i.e. $T_p(0) = 0$), so we take the minus one. So

$$T_p(x) = \frac{1 - \sqrt{1 - 4x}}{2} = \frac{1}{2} - \frac{1}{2} \sum_n \binom{1/2}{n} (-4x)^n.$$